# WIDTH OF LINEAR GROUPS

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Let $S$ be a symmetric ($S = S^{-1}$) generating set of a group $G$. Width of $G$ with respect to $S$ is the diameter of the Cayley graph of $G$ with respect to $S$. If $X$ is a subset of $G$, then the width of $X$ is the smallest integer $L$ (or infinity) such that any element of $X$ decomposes in a product of at most $L$ generators. Y.Shalom has observed that the width of $G$ (if it is finite) can be used to estimate the Kazhdan constant of $G$.

The main concern of the talk is the question: "When $S$-width of $X$ is finite?" for certain $S$ and $X$ in a Chevalley group $G$.

Let $R$ be a commutative ring and $n \geq 3$. First, we consider the width of $\mathrm{SL}_n(R)$ with respect to the set $S$ of all elementary transvections. Denote by $\mathrm{E}_n(R)$ the subgroup of $\mathrm{SL}_n(R)$ generated by $S$. In general $\mathrm{E}_n(R)$ can be a proper subgroup. But for lower-dimensional rings and polynomial rings it coincides with the whole group. If the stable rank of $R$ equals to 1, the width can be computed using Gauss decomposition.

By results of D.Carter, G.Keller we know that the width of $\mathrm{SL}_n(R)$ is finite if $R$ is a ring of integers in an algebraic number field. On the other hand, van der Kallen proved that $\mathrm{SL}_n(F[x])$ has infinite width if the transcendence degree of $F$ over its prime subfield is infinite.

**Problem.** *Does $\mathrm{SL}_n(R)$ have finite width with respect to $S$ for $R = \mathbb{F}_q[x_1, \ldots, x_m]$? $R = \mathbb{Z}[x_1, \ldots, x_m]$? $R = \mathbb{Q}[x_1, \ldots, x_m]$?*

Now, let $C$ be the set all commutators. Since $\mathrm{E}_n(R)$ is perfect for all $n \geq 3$, the set $C$ generates $\mathrm{E}_n(R)$. K.Dennis and L.Vaserstein showed that the width of stable elementary subgroup $\mathrm{E}(R)$ with respect to $C$ equals 2. In the same paper they proved that for $\mathrm{E}_n(F[x])$ ($= \mathrm{SL}_n(F[x])$) the width in commutators is infinite under conditions of van der Kallen's theorem.

I proved that given a Chevalley-Demazure group scheme $G$ (e.g. $G = \mathrm{SL}_n$) there exists a constant $L = L(G)$ such that for any ring $R$ the width of the set $C$ in elementary generators $\leq L$. The idea of the proof will be presented.

Van der Kallen noticed that the group $\mathrm{E}_n(R)^\infty / \mathrm{E}_n(R^\infty)$ is an obstruction for the bounded generation of $\mathrm{E}_n(R)$ by elementary generators, were infinite power means the direct product of countably many copies of a ring or a group.

My result above is equivalent to the fact that the obstruction group is central in $\mathrm{SK}_{1,n}(R^\infty)$, so one can study it using homological algebra.