SANDWICH CLASSIFICATION IN LINEAR GROUPS

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1. Subgroups containing a given subgroup

Let D be a subgroup of a group G. We study the lattice $\mathcal{L}$ of subgroups of G, containing D.

For a subgroup $H \leqslant G$ denote by D^H the smallest subgroup, containing D and normalized by H. The normalizer of H in G is denoted by $N_G(H)$. A subgroup $H \in \mathcal{L}$ is called D-full if $D^H = H$.

Definition. We say that the lattice $\mathcal{L}$ satisfies sandwich classification if for each subgroup $H \in \mathcal{L}$ there exists a unique D-full subgroup F such that

$$F \leqslant H \leqslant N_G(F).$$

In other words, $\mathcal{L}$ satisfies sandwich classification if for any $H \in \mathcal{L}$ the subgroup D^H is D-full.

Clearly, sandwich classification holds if D is normal in G or D is a maximal subgroup. More generally, if D is pronormal in G, then $\mathcal{L}$ satisfies sandwich classification.

2. Examples

Let $G = \mathrm{GL}_n(R)$ be the general linear group over a commutative ring R. For the following situations the lattice $\mathcal{L}$ is proved to satisfy sandwich classification.

1. D is the group of diagonal matrices, R is a field, containing at least 7 elements.
2. D is a split classical group in usual representation.
3. D is a block-diagonal group with dimensions of diagonal blocks $\geqslant 3$.

4.1. $D = \mathrm{SL}_n(K)$, where K is a Dedekind domain and R is its field of fractions.

4.2. $D = \mathrm{SL}_n(K)$, where K is a field and R is its algebraic extension.

5. $D = \mathrm{SL}_m(R) \otimes \mathrm{SL}_k(R)$, where $mk = n$, $m - 2 \geqslant k \geqslant 3$.

In all cases we have explicit description of the set of D-full subgroups.

Item 1 generalizes for GL_n theorem by Borel and Tits on closed subgroups of a reductive algebraic group over a closed field, containing maximal torus. Theorems corresponding to items 2–5 are wide generalizations of results on maximality of subgroups from corresponding Aschbacher classes in GL_n over a finite field.

Similar results (some of them conjecturally) can be stated for a Chevalley group $G = \mathrm{G}(\Phi, R)$.

3. Elementary subgroup

A subgroup of $\mathrm{GL}_n(R)$ generated by all elementary transformations $t_{ij}(r)$ is called the elementary subgroup. It is denoted by $\mathrm{E}_n(R)$. Since $[t_{ij}(r), t_{jk}(s)] = t_{ik}(rs)$, the group $\mathrm{E}_n(R)$ is perfect. Similarly, one defines the elementary subgroup of a Chevalley group. Note that in examples 2–5 above we can take elementary subgroup D (e.g. in item 4 $D = \mathrm{E}_n(K)$).

4. Subgroups, normalized by D

Now, let us consider the lattice $\mathcal{L}'$ of subgroups of G, normalized by D. Denote $[D,H]$ the mutual commutator subgroup. A subgroup $H\in\mathcal{L}'$ is called D-perfect if $[D,H]=H$. For $H\in\mathcal{L}'$ denote by $C(H)=C_{D,G}(H)$ the largest subgroup of $N_G(H)$ satisfying $[C(H),D]\leqslant H$.

Definition. We say that the lattice $\mathcal{L}'$ satisfies sandwich classification if for each subgroup $H\in\mathcal{L}'$ there exists a unique D-perfect subgroup F such that

$$F\leqslant H\leqslant C_{D,G}(F).$$

In other words, $\mathcal{L}'$ satisfies sandwich classification if for any $H\in\mathcal{L}$ we have $\big[[H,D],D\big]=[H,D]$.

Theorem 1. *Let D be a perfect subgroup (i.e. $[D,D]=D$) of a group G. Suppose that sandwich classification holds for subgroups, containing D. Then sandwich classification holds for subgroups, normalized by D.*

Denote by $\mathcal{F}$ the set of all D-full subgroups of G and for $F\in\mathcal{F}$ denote by $\mathcal{P}_F$ the set of all F-perfect subgroups of F. Then the set of all D-perfect subgroups is a union of $\mathcal{P}_F$ over all $F\in\mathcal{F}$.

5. Subgroups of $\mathrm{GL}_n(R)$, normalized by $\mathrm{E}_n(R)$

Let I be an ideal of R. Denote by $E_n(I)$ the subgroup of $\mathrm{GL}_n(R)$ generated by all elementary transfomations $t_{ij}(r)$, where $i\neq j$, $r\in I$. Let $\mathrm{E}_n(R,I)$ be the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$. Denote by $\mathrm{C}_n(R,I)$ the full congruence subgroup, i.e. the preimage of the center under the canonical homomorphism $\mathrm{GL}_n(R)\to\mathrm{GL}_n(R/I)$.

Theorem 2. *Let R be a commutative ring and $n\geqslant 3$. Given a subgroup H of $\mathrm{GL}_n(R)$, normalized by $\mathrm{E}_n(R)$ there exists a unique ideal I such that*

$$\mathrm{E}_n(R,I)\leqslant H\leqslant \mathrm{C}_n(R,I).$$

Moreover, the groups $\mathrm{E}_n(R,I)$ are the only $\mathrm{E}_n(R)$-perfect subgroups and $\mathrm{C}_n(R,I)=C\big(\mathrm{E}_n(R,I)\big)$.

6. Applications of Theorem 1 to examples 2–5

It is easy to apply Theorems 1 and 2 to examples 2 and 4. I'll discuss a conjecture for example 3 as well.