MULTIPLE RELATIVE COMMUTATOR FORMULA

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Let R be a commutative ring and $n \geq 3$. The starting point of our research is a celebrated Suslin's theorem asserted that the elementary subgroup is normal in the general linear group of order n over R. After that Vaserstein and independently Borewich and Vavilov proved the standard commutator formulae:

$$[\mathrm{GL}_n(R), \mathrm{E}_n(R, I)] = \mathrm{E}_n(R, I)$$
$$[\mathrm{E}_n(R), \mathrm{GL}_n(R, I)] = \mathrm{E}_n(R, I)$$

for any I of R. A common generalisation of both formulae above is a relative commutator formula

$$[\mathrm{E}_n(R, I), \mathrm{GL}_n(R, J)] = [\mathrm{E}_n(R, I), \mathrm{E}_n(R, J)]$$

where I, J are ideals in R (it is easy to show that $[\mathrm{E}_n(R, I), \mathrm{E}_n(R)] = \mathrm{E}_n(R, I)$). If $I + J = R$, then $[\mathrm{E}_n(R, I), \mathrm{E}_n(R, J)] = \mathrm{E}_n(R, IJ)$, otherwise there are counterexamples to this equation (A.Mason).

The subject of the talk is the proof of multiple relative commutator formula

$$\begin{aligned}[\mathrm{E}_n(R, I_1), \mathrm{GL}_n(R, I_2), \ldots, \mathrm{GL}_n(R, I_m)] &= [\mathrm{E}_n(R, I_1), \mathrm{E}_n(R, I_2), \ldots, \mathrm{E}_n(R, I_m)] \\ &= [\mathrm{E}_n(R, I_1 I_2 \ldots I_{m-1}), \mathrm{E}_n(R, I_m)]\end{aligned}$$

which generalises all of the above. It was recently proved by R.Hazrat and Z.Zhang using "yoga of commutators".

I shall show the proof of the multiple relative commutator formula without any calculations. The proof is based on the Quillen–Suslin localisation method and one of the main concern of the talk is a statement of the localisation principle in general settings, in categorical terms.