BOUNDED GENERATION OF LINEAR GROUPS

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Width of a group G with respect to a generating set S is the diameter of the Cayley graph of G with respect to S. In other words, the width is the smallest integer L (or infinity) such that any element of G decomposes in a product of at most L generators.

Let $G = \mathrm{SL}_n(R)$ where R is a commutative ring. Put $t_{ij}(r) = e + re_{ij}$, where $r \in R$, e denotes the identity matrix and e_{ij} is the standard matrix unit, i.e. the matrix with 1 in position (i, j) and zeroes elsewhere. Note that (left) multiplication by $t_{ij}(r)$ is an elementary Gauss transformation. Therefore, the set of all $t_{ij}(r)$ is called the set of elemetary generators. By Gauss algorithm we know that over a field elementary generators span G and the width of G equals to $O(n^2)$.

Over general rings both facts are false. Denote by $\mathrm{E}_n(R)$ a subgroup of G spanned by the elementary generators. First, it is not even obvious, that $\mathrm{E}_n(R)$ is normal in G and for noncommutative rings this fails in general (example of V.Gerasimov).

Theorem (A.Suslin)**.** *If R is commutative and $n \geq 3$, then $\mathrm{E}_n(R)$ is normal in $\mathrm{SL}_n(R)$.*

The quotient group $\mathrm{SL}_n(R)/\,\mathrm{E}_n(R)$ is denoted by $\mathrm{SK}_{1,n}(R)$. If we replace SL by GL and pass to the limit on $n \to \infty$, we get the Whitehead group $\mathrm{K}_1(R)$ which fits into an extraordinary cohomology sequence of K-functors. The group $\mathrm{SK}_{1,n}(R)$ can even be non-Abelian (A.Bak, W. van der Kallen).

Theorem (A.Bak)**.** *Let R be a commutative ring, $\dim \operatorname{Max} R = d < \infty$, and $n \geq 3$. Then $\mathrm{SK}_{1,n}(R)$ is nilpotent of nilpotent class at most $d + 1$.*

For low-dimensional rings and for polynomial rings the natural map $\mathrm{SK}_{1,n}(R) \to \mathrm{SK}_1(R)$ is an isomorphism.

Theorem (H.Bass, L.Vaserstein)**.** *Let R be a commutative ring, $\dim \operatorname{Max} R = d \leq n - 2$. Then the natural map $\mathrm{SK}_{1,n}(R) \to \mathrm{SK}_1(R)$ is an isomorphism.*

The following theorem is a key step in the solution of congruence subgroup problem.

Theorem (H.Bass, J.Milnor, J.-P.Serre)**.** *Let R be a ring of integers in an algebraic number field. Then $\mathrm{SK}_1(R)$ is trivial.*

The following result is called K_1-analogue of Serre's problem.

Theorem (A.Suslin)**.** *Let R be a ring such that $\mathrm{SK}_1(R)$ is trivial. Then $\mathrm{SK}_1(R[x_1, \dots, x_m])$ is also trivial.*

A group G is *boundedly generated* by a subset $S \subseteq G$ if the width of G with respect to S is finite. Y.Shalom has observed that bounded generation of $G = \mathrm{SL}_n(R)$ can be used to estimate the Kazhdan constant of G.

Theorem (D.Carter, G.Keller)**.** *Let R be a ring of integers in an algebraic number field and $n \geq 3$. Then $\mathrm{SL}_n(R)$ is boundedly generated by the elementary generators.*

Theorem (W. van der Kallen)**.** *Let F be a field. If the transcendence degree of F over its prime subfield is infinite, then $\mathrm{SL}_n(F[x])$ is not boundedly generated by the elementary generators.*

Problem. *Is* $\mathrm{SL}_n(R)$ *boundedly generated by elementary generators for* $R = \mathbb{F}_q[x_1, \dots, x_m]$, $R = \mathbb{Z}[x_1, \dots, x_m]$, $R = \mathbb{Q}[x_1, \dots, x_m]$ *? (the problem is unsolved already for* $m = 1$*).*

Now, let C be the set all commutators. Since $\mathrm{E}_n(R)$ is perfect for all $n \geq 3$, the set C generates $\mathrm{E}_n(R)$. K.Dennis and L.Vaserstein showed that the width of stable elementary subgroup $\mathrm{E}(R)$ with respect to C equals 2. In the same paper they proved that for $\mathrm{E}_n(F[x])$ $(= \mathrm{SL}_n(F[x])\,)$ the width is infinite under conditions of van der Kallen's theorem.

Theorem (Stepanov)**.** *Let* R *be a commutative ring,* $n \geq 3$*, and* $G = \mathrm{SL}_n(R)$*. Then there exists a constant* $L = L(n)$ *such that for any elements* $a \in G$ *and* $b \in \mathrm{E}(R)$ *the commutator* $[a, b]$ *is a product of at most* L *elementary generators.*

Van der Kallen noticed that the group $\mathrm{E}_n(R)^\infty / \mathrm{E}_n(R^\infty)$ is an obstruction for the bounded generation of $\mathrm{E}_n(R)$ by elementary generators, were infinite power means the direct product of countably many copies of a ring or a group.

The theorem above is equivalent to the fact that this group is central in $\mathrm{SK}_{1,n}(R^\infty)$, so one can study it using homological algebra.