OVERGROUPS OF SUBRING SUBGROUPS

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All rings are assumed to be commutative with 1. Let K be a ring, let $S \subseteq A$ be K-algebras and G an algebraic group scheme over K. This is a well known problem: to describe lattice of subgroups between $G(S)$ and $G(A)$.

The talk is about this problem for a Chevalley–Demazure group scheme $G = \mathrm{G}(\Phi, \_)$ with a root system $\Phi \neq A_1$ (we can always asume that $K = \mathbb{Z}$). For a ring R let $E(R) = \mathrm{E}(\Phi, R)$ denote the elementary subgroup of $G(R)$. Then the answer is slightly easier to formulate for the lattice of subgroups between $E(S)$ and $G(A)$.

The standard description of this lattice is called standard sandwich calssification (SSC).

Definition. Fix a triple (Φ, S, A). The SSC holds if given a subgroup H between $E(S)$ and $G(A)$ there exists a unique subring R between S and A such that

$$E(R) \leqslant H \leqslant N_A(R)$$

where $N_A(R)$ denotes the normalizer of $E(R)$ in $G(A)$.

Known results. SSC holds:

0. $A = S$ and any Φ. In this case SSC follows from normality of $E(S)$ in $G(S)$ (Taddei, 1986).
1. $\Phi = A_n$, A is the field of fractions of a Dedekind domain S (R.A.Shmidt, 1979).
2. Any $\Phi \neq A_1$, A is the field of fractions of a PID S (Nuzhin, Yakushevich, 2000).
3. Any $\Phi \neq A_1$, A is an algebraic extension of a field S (Nuzhin, 1983).

SSC does not hold:

4. Φ is simply laced (i.e. $\Phi = A_n, D_n, E_n$), arbitrary S, A is the affine algebra of G over S (follows easily from Gordeev's theorem published in 1998)

I present the followng new results.

Theorem 1. *Let Φ be doubly laced, i. e. $\Phi = B_n, C_n, F_4$, where $n \geqslant 2$, and $\frac{1}{2} \in S$. Suppose further that -1 is a square in S if $\Phi = B_n$. Then SSC holds for all pairs $S \subseteq A$.*

In contrast, for simply laced root systems the situation is dramatically different.

Theorem 2. *Suppose that Φ is simply laced, S is a field, $A = S[t]$ and G is of adjoint type. Then there exists a nontrivial element $g \in E(F[t])$ such that the subgroup generated by $E(S)$ and g is the free product of $E(F)$ and the cyclic subgroup generated by g.*

It is intuitively clear that nothing like SSC can hold in a free product. To formulate the corollary of the result above we need the following notion.

Definition. A ring A is called *quasi transcendental* over its subring S if there exists a diagram

$$\begin{array}{ccccc} S & \longrightarrow & P & \longrightarrow & A \\ \downarrow & & \downarrow & & \\ F & \longrightarrow & F[t] & & \end{array}$$

where F is a field, horizontal arrows are natural inclusions, and vertical ones are surjective. Otherwise, A is called *quasi algebraic* over S.

The condition of being quasi algebraic has a simple reformulation for S of geometric origin. Suppose that A is a domain and denote by F the algebraic closure of the field of fractions of S.

Theorem 3. *Suppose that S is a finitely generated algebra over a field or over $\mathbb{Z}$, and A is a domain. Then A is quasi algebraic over S iff either A is an integral extension of S or* $\dim S \leqslant 1$ *and A embeds into F.*

Corollary 1. *Suppose that Φ is simply laced. If A is quasi transcendental over S then SSC does not hold.*