STRUCTURE OF CHEVALLEY GROUPS OVER COMMUTATIVE RINGS

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Definition of a Chevalley group (= group of rational points of a Chevalley–Demazure group scheme) would take too much time. Instead of giving the definition I shall briefly explain main notions (split maximal torus, Borel subgroup, Bruhat decomposition, root subgroups, root system) by example of the special linear group.

After that I shall define the elementary subgroup, relative elementary subgroup and congruence subgroups.

We always assume that $\Phi \neq A_1$. Moreover, for simplicity we shall assume that 2 is invertible for $\Phi = B_l, C_l, F_4$ and 6 is invertible for $\Phi = G_2$.

Theorem 1 (Standard commutator formulas)**.** *Let I be an ideal in R. Then*

$$[\mathrm{E}(\Phi, R, I), \mathrm{G}(\Phi, R)] = [\mathrm{E}(\Phi, R), \mathrm{C}(\Phi, R, I)] = \mathrm{E}(\Phi, R, I).$$

Theorem 2 (Normal structure theorem)**.** *Given a subgroup H of $\mathrm{G}(\Phi, R)$, normalized by $\mathrm{E}(\Phi, R)$, there exists a unique ideal I such that*

$$\mathrm{E}(\Phi, R, I) \leqslant H \leqslant \mathrm{C}(\Phi, R, I).$$

It follows that the lattice $\mathcal{L}$ of subgroups of $G(\Phi, R)$, normalized by $\mathrm{E}(\Phi, R)$, splits into disjoint union of "sandwiches" $\mathrm{E}(\Phi, R, I) \leqslant \cdots \leqslant \mathrm{C}(\Phi, R, I)$. Since the group $\mathrm{C}(\Phi, R, I)/\,\mathrm{G}(\Phi, R, I)$ is well understood (it is isomorphic to the center of $\mathrm{G}(\Phi, R/I)$ which is a subgroup of $(R/I)^*$), structure of $\mathcal{L}$ is determined by the structure of the groups $\mathrm{K}_1(\Phi, R, I) = \mathrm{G}(\Phi, R, I)/\,\mathrm{E}(\Phi, R, I)$.

Theorem 3 (Nilpotent structure of K_1)**.** *Let R be a commutative ring. Denote by d the dimension of maximal spectrum of R. If G is simply connected, then $\mathrm{K}_1(\Phi, R, I)$ is a nilpotent group of nilpotent degree at most $d+1$. In general, $\mathrm{K}_1(\Phi, R, I)$ is an extension of a nilpotent group of nilpotent degree at most $d+1$ by an Abelian group (in fact the structure of this Abelian group also is well understood).*

Another important problem is a structure of the elementary subgroup itself. The elementary subgroup is defined by its generators. So one can hope to find a complete set of relations between them. The following relations are considered as "trivial":

$$x_\alpha(r)x_\alpha(s) = x_\alpha(r+s); \qquad [x_\alpha(r), x_\beta(s)] = \prod_{i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}r^i s^j),$$

for all $\alpha, \beta \in \Phi$ and $r, s \in R$. The first relation follows from the definition of a root subgroup, the second is called Chevalley commutator formula.

The group generated by *symbols* $y_\alpha(r)$, $\alpha \in \Phi$, $r \in R$, subject to relations

$$y_\alpha(r)y_\alpha(s) = y_\alpha(r+s); \qquad [y_\alpha(r), y_\beta(s)] = \prod_{i\alpha+j\beta\in\Phi} y_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}r^i s^j),$$

for all $\alpha, \beta \in \Phi$ and $r, s \in R$, is called the Steinberg group and denoted by $\mathrm{St}(\Phi, R)$. Now, the set of "nontrivial" relations between elementary generators are mesured by the group $\mathrm{K}_2(\Phi, R) = \mathrm{Ker}\big(\mathrm{St}(\Phi, R) \to E(\Phi, R)\big)$.

Theorem 4 (centrality of K_2)**.** *Let* $\Phi = A_l$*,* $l \geqslant 3$ *(i.e.* $\mathrm{G}(\Phi, R) = \mathrm{SL}_n(R)$*,* $n \geqslant 4$*). Then* $\mathrm{K}_2(n, R)$ *lies in the center of the Steinberg group* $\mathrm{St}_n(R)$*. Moreover, if* $n \geqslant 5$*, then* $\mathrm{St}_n(R)$ *is a universal central extension of* $\mathrm{E}_n(R)$*, hence* $\mathrm{K}_2(n, R)$ *is the second cohomology group of* $\mathrm{E}_n(R)$*.*

Even for classical groups centrality of K_2 has not been proved yet (the proof for general unitary groups, which is a common generalization of all classical groups, was announced by A.Bak and G.Tang more than 10 years ago but an actual proof has not appeared yet).

Centrality of $\mathrm{K}_2(3, R)$ still is an open question.