Var. 1 (131106) *Adeel*

1. Find all possible Jordan forms of a matrix A, satisfying the equation $A^2 = A^3$.

2. Given square matrices A and B over a field of characteristic 0, satisfying the equation $AB - BA = B$ prove that 0 is the only eigenvalue of B.

Hint: To begin with, consider the case where B is a Jordan block.

Give a counterexample to the above statement in a prime characteristic.

Var. 2 (131106) *Ali Ovais*

1. Given a matrix A with the entries $a_{i,n+1-i} = \alpha_i \in \mathbb{C}$ and all other zeroes find a condition on α_i's which is equivalent to diagonalizability of A over $\mathbb{C}$?

2. Let F be an algebraically closed field. Let M be a subset in $M_n(F)$ consisting of pairwise commuting matrices. Prove that there exists a common eigenvector of all matrices from M.

Var. 3 (131106) *Faraha*

1. Prove that over $\mathbb{C}$ any periodic matrix is diagonalizable (a matrix A is called periodic if $A^m = A$, for some $m \geqslant 2$). For which primes p the same statement holds over an algebraically closed field of characteristic p?

Hint. Since $\binom{p}{k}$ is divisible by p for all $1 \leqslant k \leqslant p-1$, $(a+b)^p = a^p + b^p$ in a commutative algebra over a field of characterstic p.

2. Let $A \in \mathrm{M}_n(\mathbb{C})$ and $B \in \mathrm{M}_k(\mathbb{C})$ be matrices without common eigenvalues. Prove that the matrix equation $AX = XB$, where $X \in \mathrm{M}_{n\times k}(\mathbb{C})$ have no nonzero solutions.

Hint. Construct a basis of $\mathrm{M}_{n\times k}(\mathbb{C})$ out of Jordan bases of A and B and write the matrix of the operator $L(X) = AX - XB$ in this basis.

Var. 4 (131106) *Kamran*

1. Let V be a vector space over a field K and let f be a basis of V. Given a linear operator $L : V \to V$ such that

$$L_f = \begin{pmatrix} \alpha_1 & 1 & 0 & 0 & \dots & 0 \\ \alpha_2 & 0 & 1 & 0 & \dots & 0 \\ \alpha_3 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & 0 & 0 & 0 & \dots & 1 \\ \alpha_n & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

prove that it has a nontrivial invariant subspace iff the polynomial

$$x^n - a_1 x^{n-1} - \dots - a_{n-1}x - a_n$$

is reducible over K.

2. Let $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ and $B \in \mathrm{M}_{n\times m}(\mathbb{R})$ where $m \leqslant n$. Given a characteristic polynomial of AB determine the characteristic polynomial of BA.

Hint. First, consider the case where $n = m$ and A is invertible. Then, using topological arguments show that the same holds for arbitrary square matrices. Finaly, deduce the general case from the above.

Var. 5 (131106) *Ahsan Khan*

1. Suppose that a matrix $A \in \mathrm{M}_n(\mathbb{C})$ has a single eigenvalue and its geometric multiplicity equals 1. Describe the set of all matrices commuting with A (prove that this set is a subspace of $\mathrm{M}_n(\mathbb{C})$). In particular, find the dimension of this subspace.

2. Prove that for any square matrix A over an algebraically closed field there exists a matrix C such that $A^T = C^{-1}AC$.

Var. 6 (131106) *Yameen*

1. Prove that a matrix $A \in \mathrm{M}_n(\mathbb{C})$ is nilpotent (i. e. $A^k = 0$ for some positive integer k) iff 0 is the only eigenvalue of A. Determine the nilponency degree of A in terms of its Jordan form (the nilpotency degree is the smallest k such that $A^k = 0$).

2. Let V be a complex vector space over a field K and let e be a basis of V. Given a linear operator $L : V \to V$ such that $A(e_k) = e_{k+1}$ for $k < n$ and $A(e_n) = e_1$ find the Jordan form and a Jordan basis of A.

Hint. Guess the minimal polynomial of A.

Var. 7 (131106) *Nehad*

1. Suppose that an operator A in a n-dimensional complex vector space has only one 1-dimensional invariant subspace. Determine the Jordan form of the operator A^k, (where k is a positive integer).
2. Let $A \in \mathrm{M}(2, \mathbb{C})$. Complete the statement "The matrix equation $X^2 = A$ has no solutions iff the matrix $A \ldots$" and prove it.

Var. 8 (131106) *Shamas*

1. Let $A \in \mathrm{M}(n, \mathbb{C})$ has n distinct eigenvalues. Determine the number of A-invriant subspaces.
2. Let

$$M = \left\{ \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{C},\ 0 \leqslant \arg b < \pi \text{ or } b = 0 \right\} \cup \left\{ cI \mid c \in \mathbb{C} \setminus \{0\} \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

   Prove that any matrix from $\mathrm{M}_2(\mathbb{C})$ is conjugate to exactly one matrix from M.

Var. 9 (131106) *Umar*

1. Determine the Jordan form of a matrix with a on the main diagonal and b elsewhere.
2. A n-dimensional complex vector space V can be regarded as a $2n$-dimensional real vector space $\tilde{V}$ (as a set $\tilde{V} = V$). Suppose that the characteristic polynomial of an operator $A : V \to V$ is equal to $\prod_{i=1}^{n} (t - \lambda_i)$ (the numbers λ_i are not assumed to be distinct). Let $\tilde{A} : \tilde{V} \to \tilde{V}$ coincides with A as a map. Determine eigenvalues of $\tilde{A}$.

Var. 10 (131106) *Yasir*

1. Let V be a n-dimenisonal vector space. A linear operator P on V is called a *projector* if $P^2 = P$. Prove that any projector splits into a sum of projectors of rank 1 such that all pairwise products are equal to 0.
2. Suppose that an operator $A : \mathbb{C}^n \to \mathbb{C}^n$ has a single eigenvalue and its geometric multiplicity equals 1. Find all A-invariant subspaces of $\mathbb{C}^n$.

Var. 11 (131106) *Zunaira*

1. Prove that for any operator L in a finite dimensional complex vector space V there exists a basis f of V such that $L_f = L_f^T$.
   Hint. Show that a Jordan block is conjugate to a symmetric matrix by the matrix $\frac{1}{\sqrt{2}}(I + iB)$, where B is the matrix with 1 in the right-hand side diagonal and 0 elsewhere.
2. Let F be an algebraically closed field and $A \in \mathrm{M}_n(F)$. Find a condition on A equivalent to the following statement: Given a polynomial $p \in F[t]$ there exists a common Jordan basis for A and $p(A)$.