The NIM game

Problem. A position is a bunch of several rows of matches. Two players make moves in turn. During his move a player chooses a row and takes as much matches from this row as he wants. The player taking last match wins. The problem is to describe positions in which the first player loose, assuming both sides make optimal moves.

Gidelines. A position with a_i matches in i-th row ($i = 1, \dots, n$) will be denoted by $(a_1, \dots, a_n)$. Clearly, play does not depend on adding or erasing empty rows and on permutation of rows. Thus, we shall identify positions which differs only by zero elements and by a permutation.

A position in which the first player loose assuming both sides make optimal moves will be called a *loosing position.*

Lemma 1. *Position* (a, b) *is loosing iff* $a = b$.

Lemma 2. *Given* $a_1, \dots, a_n \in \mathbb{N}$ *there exists a unique* c *such that position* $(a_1, \dots, a_n, c)$ *is loosing.*

By the previous lemma we can define an operation $\oplus$ on $\mathbb{N}$ by the rule: $a \oplus b = c$ iff (a, b, c) is a loosing position. Define also an operation $\perp$ (orthogonal sum) on the set of positions as concatenation, i. e.
$(a_1, \dots, a_n) \perp (b_1, \dots, b_n) = (a_1, \dots, a_n, b_1, \dots, b_n)$.

Lemma 3. *Let* a *be a loosing position. A position* b *is loosing iff position* $a \perp b$ *is loosing.*

Lemma 4.

(1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(2) *A position* $(a_1, \dots, a_n)$ *is loosing iff* $a_1 \oplus \cdots \oplus a_n = 0$.

Corollary 5. *The set* $\mathbb{N}$ *is a vector space over* $\mathbb{F}_2$ *with respect to addition* $\oplus$ *and obvious multiplication by scalars. It will be denoted by* $\mathbb{N}^{\oplus}$.

Lemma 6. $a \oplus b \leq a + b$

Lemma 7. *Elements* $1, 2, 4, \dots, 2^m, \dots$ *form a basis for* $\mathbb{N}^{\oplus}$. *Moreover,* $2^m \oplus n = 2^m + n$ *porvided* $n < 2^m$.

The last lemma allows us to find the coordinates of a natural number in the chosen basis and, thus, to find the rule of performing $\oplus$-operation. By Lemma 4 this is sufficient for describing loosing positions.