Problem. Let A and B be n by n complex matrices. Assume that their ring-theoretical commutator $AB - BA$ has rank 1. Prove that they have a common eigenvector.

Gidelines.

Lemma 1. *A matrix of rank 1 can be represented as a product of a column and a row.*

By terminological reasons we shall identify operators on Hermitian space $\mathbb{C}^n$ with their matrices in the standard basis.

Lemma 2. *Without loss of generality we may replace* A *by* $A - \lambda I$ *and, thus, we may assume that the operator* A *has a nonzero kernel.*

Lemma 3. $(AB)^* = B^*A^*$ *(where star denotes the dual operator).*

The next lemma is a key step in the proof.

Lemma 4. *If* $\operatorname{rank}(AB - BA) = 1$ *then either* $\operatorname{Ker} A$ *is invariant under* B *or* $\operatorname{Ker} A^*$ *is invariant under* B^**.*

Hint. Write $AB - BA = uv$ for a suitable column u and row v and multiply this equation by a column $x \in \operatorname{Ker} A$ from the right and by a row y, where $y^* \in \operatorname{Ker} A^*$, from the left (here y^* denotes transpose conjugated to y).

Lemma 5. *If* x *is an eigenvector of* A^**, then* $\langle x \rangle^\perp$ *is invariant under* A*.*

The last lemma shows that if A^* and B^* have a common eigenvector, then A and B have common invariant subspace of dimension less that n. Thus, the proof can be competed by induction on n.